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Some asymptotic properties for orthogonal polynomials with respect to varying measures☆

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Abstract

We study some properties of the zeros and the asymptotic behavior of orthogonal polynomials with respect to varying measures on the unit circle. In the proofs, some techniques of rational approximation are used.

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Keywords: Orthogonal polynomials; Varying measures; Zeros; Asymptotics

☆ This paper contains some results of the unpublished manuscript [3]. J.J. Guadalupe unexpectedly died in 2000. We want that this paper pays homage to him.

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1. Introduction and main results

Let $\mathcal{M}$ denote the set of finite positive Borel measures on $[0, 2\pi)$ with an infinite set of points in their support, and let $\mu \in \mathcal{M}$. Given a sequence of polynomials $\{W_n(z) = \prod_{j=1}^{n} (z - \omega_{n,j})\}_{n=1}^{\infty}$ with $|\omega_{n,j}| \leqslant 1$, $1 \leqslant j \leqslant n$, set

$$d\mu_n(\theta) = \frac{d\mu(\theta)}{|W_n(e^{i\theta})|^2}, \quad n \in \mathbb{N}.$$

In the sequel, we assume that μ_n is finite for all $n \in \mathbb{N}$. In particular, this is true if all the zeros of W_n are inside the unit disk for all $n \in \mathbb{N}$. Obviously, $\mu_n \in \mathcal{M}$, $n \in \mathbb{N}$. For every n, let $\{\varphi_m(\mu_n; z)\}_{m=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to μ_n. That is,

$$\varphi_m(\mu_n; z) = \kappa_m(\mu_n) z^m + \text{lower degree terms}, \quad \kappa_m(\mu_n) > 0$$

and

$$\int_0^{2\pi} \varphi_k(\mu_n; z) \overline{\varphi_m(\mu_n; z)}\, d\mu_n(\theta) = \begin{cases} 0, & k \neq m, \\ 1, & k = m, \end{cases} \quad z = e^{i\theta}.$$

Set $\Phi_m(\mu_n; z) = \frac{\varphi_m(\mu_n; z)}{\kappa_m(\mu_n)}$ and $\Phi_m^*(\mu_n; z) = z^m \overline{\Phi\left(\mu_n; 1/\overline{z}\right)}$.

If $\omega_{n,j} = 0$, $j = 1, 2, \ldots, n$, then $|W_n(e^{i\theta})| = 1$, $\theta \in [0, 2\pi)$, and the orthogonal polynomials with respect to these varying measures become the orthogonal polynomials with respect to the fixed measure μ.

Another case of particular interest arises when we take $W_n(z) = \varphi_n(\mu; z)$ and the varying weight is $d\theta/|\varphi_n(\mu; z)|^2$. Using the Geronimus identity (see [7, p. 198, formula (2.2)])

$$\int \frac{z^j}{|\varphi_n(\mu; z)|^2}\, d\theta = \int z^j\, d\mu, \quad j = 0, \pm 1, \ldots, \pm n, \ \ z = e^{i\theta},$$

it follows that $\varphi_m(d\theta/|\varphi_n(\mu; \cdot)|^2; z) = \varphi_m(\mu; z)$, $m = 0, 1, \ldots, n$. Obviously, this formula can be written in a more general way, namely

$$\int \frac{z^j}{|\varphi_m(\mu_n; z)|^2}\, d\theta = \int z^j\, d\mu_n, \quad j = 0, \pm 1, \ldots, \pm m, \ \ z = e^{i\theta}; \tag{1}$$

this expression will be useful in some places in this paper.

Orthonormal polynomials with respect to varying measures were introduced about 25 years ago by A.A. Gonchar and G. López in connection with a systematic study of the convergence properties of interpolating rational functions with free poles to Markov functions. In [10], López presents orthogonal polynomials with respect to varying measures in such a way that unifies the theory for the cases of measures with bounded and unbounded support. That paper also shows that orthogonal polynomials with respect to varying measures are a powerful tool in solving problems where a fixed measure and orthogonality in the usual sense are involved. Other applications in that direction can be found in [4,5].

This paper focuses on two goals. First, we study the asymptotic behavior of orthogonal polynomials with respect to varying measures. Polynomial approximation is an effective instrument in finding properties of orthogonal polynomials with respect to fixed weights (see [13,14]). In contrast, we use rational approximation to obtain the following result on the asymptotic behavior of orthogonal polynomials with respect to varying measures. We use the standard notation $\|f\|_{L^p(\mu)} = (\int_0^{2\pi} |f|^p \, d\mu)^{1/p}$ and, as usual, the Lebesgue–Radon decomposition $d\mu = \mu' d\theta + d\mu_s$. We suppose that $\mu' = +\infty$ on the support of μ_s.

Theorem 1. *If $\mu' > 0$ a.e. on $[0, 2\pi)$, then*

$$\left\| \left| \frac{\varphi_n(\mu_n; \cdot)}{W_n} \right|^2 - \frac{1}{\mu'} \right\|_{L^1(\mu)} \leqslant 2 \min \left\{ \left\| \left| \frac{p_n}{W_n} \right| - \frac{1}{\sqrt{\mu'}} \right\|_{L^2(\mu)} : p_n \in \Pi_n \right\}, \tag{2}$$

where Π_n denotes the set of polynomials of degree at most n. Moreover, if

$$\lim_{n\to\infty} \frac{W_n(z)}{W_n^*(z)} = 0 \tag{3}$$

uniformly on compact subsets of $\{z : |z| < 1\}$, then

$$\lim_{n\to\infty} \min \left\{ \left\| \left| \frac{p_n}{W_n} \right| - \frac{1}{\sqrt{\mu'}} \right\|_{L^2(\mu)} : p_n \in \Pi_n \right\} = 0. \tag{4}$$

Remark 1. An alternative sufficient condition for (4) is $\lim_{n\to\infty} \sum_{j=1}^{n} (1 - |\omega_{n,j}|) = \infty$, since this implies (3) (see [17, Theorem 9, p. 247]).

For fixed measures, a formula similar to (2) allowed Nevai and Totik to give in [13] a simple proof of Denisov's theorem [6]. Following step by step the proof in [13], using Theorem 1, and the method of varying measures employed in [4], we get Denisov's theorem on an arc (see [16, Theorem 13.4.4] for an alternative proof).

It is known that monic orthogonal polynomials with respect to a fixed measure on the unit circle are completely determined by some of their zeros (see [2]). In the next theorem we give a bound of the n-root limit of the reflection coefficients of such orthogonal polynomials; this bound is given in terms of the zeros of the orthogonal polynomials.

Theorem 2. *Suppose that $\Phi_n(\mu; z)$ has a zero in $D_r = \{z : |z| \leqslant r\}$ for every $n \in \mathbb{N}$ sufficiently large. Then*

$$\limsup_{n\to\infty} |\Phi_n(\mu; 0)|^{1/n} \leqslant \min\{1, \ 2r\}. \tag{5}$$

The paper is organized as follows. In Section 2 we prove the two main results just stated. Then, in Section 3 we give some results on varying measures related to Theorem 1. In Section 4 we prove some corollaries of Theorem 2 dealing with zeros of orthogonal polynomials; in particular, we show how to answer a question posed in [16]. Finally, we give some numerical experiments that indicate that the estimate in (5) seems to be sharp.

2. Proof of the main results

2.1. Proof of the Theorem 1

Let us begin with the following lemma, which is a small variation of Walsh [17, Corollary 2, p. 246]:

Lemma 1. *Let* $\{W_n(z)\}_{n=1}^{\infty}$ *be a sequence of polynomials as indicated above. The following conditions are equivalent*:

(a) $\lim_{n\to\infty} W_n(z)/W_n^*(z) = 0$ *uniformly on compact subsets of* $\{z : |z| < 1\}$.
(b) *For all* f *holomorphic on* $D_1 = \{z : |z| \leqslant 1\}$, *there exists a sequence of polynomials* $\{p_n(z)\}_{n=1}^{\infty}$ *such that* $\lim_{n\to\infty} p_n(z)/W_n^*(z) = f(z)$ *uniformly on* D_1.

Let us also state another auxiliary result. Although it seems to be well-known, we could not find reference to its explicit proof, so, for completeness, we include it here.

Lemma 2. *Assume that* $\lim_{n\to\infty} W_n(z)/W_n^*(z) = 0$ *uniformly on compact subsets of* $\{z : |z| < 1\}$. *Then, for every continuous function* f *on* $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$, *there exist two sequences of polynomials* $\{p_n(z)\}_{n=1}^{\infty}$, $\{q_n(z)\}_{n=1}^{\infty}$ *with* $\deg p_n(z) \leqslant n$, $\deg q_n(z) \leqslant n$, *such that*

$$\lim_{n\to\infty} \max\left\{\left|f(z) - \frac{p_n(z) + q_n(\frac{1}{z})}{|W_n(z)|^2}\right| : z \in \Gamma\right\} = 0. \tag{6}$$

Moreover, *if* f *is nonnegative on* Γ *we can find polynomials* $t_n(z)$, $n \in \mathbb{N}$, *such that*

$$\lim_{n\to\infty} \max\left\{\left|f(z) - \left|\frac{t_n(z)}{W_n(z)}\right|^2\right| : z \in \Gamma\right\} = 0. \tag{7}$$

Proof. Let $\varepsilon > 0$ and f be a continuous function on Γ. From Weierstrass' theorem on the approximation of continuous functions by trigonometric polynomials (for example, see [17, p. 38]), we know that there exist polynomials $s(z)$, $t(z)$ such that

$$\max\{|f(z) - (s(z) + t(1/z))| : z \in \Gamma\} < \frac{\varepsilon}{2}. \tag{8}$$

From Lemma 1, we can find a natural number N_ε and two sequences of rational functions $\{\frac{p_{n,1}}{W_n^*}\}$, $\{\frac{q_{n,1}}{W_n^*}\}$ such that, for all $n \geqslant N_\varepsilon$,

$$\max\left\{\left|s(z) - \frac{p_{n,1}(z)}{W_n^*(z)}\right| : z \in \Gamma\right\} < \frac{\varepsilon}{4}, \quad \max\left\{\left|\bar{t}(z) - \frac{q_{n,1}(z)}{W_n^*(z)}\right| : z \in \Gamma\right\} < \frac{\varepsilon}{4},$$

where $\bar{t}(z)$ denotes the polynomial whose coefficients are the conjugate of the coefficients of $t(z)$. Thus, for all $n \geqslant N_\varepsilon$,

$$\max\left\{\left|s(z) + t\left(\frac{1}{z}\right) - \frac{\overline{W_n^*}(\frac{1}{z})p_{n,1}(z) + \overline{q_{n,1}}(\frac{1}{z})W_n^*(z)}{|W_n(z)|^2}\right| : z \in \Gamma\right\} < \frac{\varepsilon}{2}. \tag{9}$$

Since $\overline{W_n^*}(\frac{1}{z})p_{n,1}(z) + \overline{q_{n,1}}(\frac{1}{z})W_n^*(z) = p_n(z) + q_n(\frac{1}{z})$, where $p_n,\ q_n$ are polynomials of degree at most *n*, and $|W_n^*(z)| = |W_n(z)|$ on Γ, (6) follows immediately from (8) and (9).

Statement (7) can be deduced from (6) and the fact that every positive trigonometric polynomial of degree *n* can be represented as $|s(z)|^2$ where *s* is an algebraic polynomial of degree *n* (see [7, p. 211]). □

It is worth remarking that the condition $\lim_{n\to\infty} \Phi_n(\mu; 0) = 0$ is equivalent to

$$\lim_{n\to\infty} \Phi_n(\mu; z)/\Phi_n^*(\mu; z) = 0$$

uniformly on compact subsets of $\{z : |z| < 1\}$. Thus, taking into account Lemma 2, such orthogonal polynomials are a good election as denominators in rational approximation of continuous functions on Γ.

We already have all the machinery to prove Theorem 1. Using that $(\mu')^{-1/p} \in L^p(\mu)$, and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\left\| \left|\frac{\varphi_n(\mu_n;\cdot)}{W_n}\right|^2 - \frac{1}{\mu'}\right\|_{L^1(\mu)} \\
&\quad\leqslant \left\| \left|\frac{\varphi_n(\mu_n;\cdot)}{W_n}\right|^2 - \frac{1}{\sqrt{\mu'}}\left|\frac{p_n}{W_n}\right|\right\|_{L^1(\mu)} + \left\| \frac{1}{\sqrt{\mu'}}\left|\frac{p_n}{W_n}\right| - \frac{1}{\mu'}\right\|_{L^1(\mu)} \\
&\quad= \left\| \left|\frac{\varphi_n(\mu_n;\cdot)}{W_n}\right| \left(\left|\frac{\varphi(\mu_n;\cdot)}{W_n}\right| - \frac{1}{\sqrt{\mu'}}\left|\frac{p_n}{\varphi_n(\mu_n;\cdot)}\right|\right)\right\|_{L^1(\mu)} \\
&\qquad + \left\| \frac{1}{\sqrt{\mu'}}\left(\left|\frac{p_n}{W_n}\right| - \frac{1}{\sqrt{\mu'}}\right)\right\|_{L^1(\mu)} \\
&\quad\leqslant \left\| \left|\frac{\varphi_n(\mu_n;\cdot)}{W_n(z)}\right| - \frac{1}{\sqrt{\mu'}}\left|\frac{p_n}{\varphi_n(\mu_n;\cdot)}\right|\right\|_{L^2(\mu)} + \left\| \left|\frac{p_n}{W_n}\right| - \frac{1}{\sqrt{\mu'}}\right\|_{L^2(\mu)}.
\end{aligned}$$

But taking (1) into account, we obtain

$$\begin{aligned}
&\left\| \left|\frac{\varphi_n(\mu_n;\cdot)}{W_n}\right| - \frac{1}{\sqrt{\mu'}}\left|\frac{p_n}{\varphi_n(\mu_n;\cdot)}\right|\right\|^2_{L^2(\mu)} \\
&\quad = 1 - 2\int_0^{2\pi} \left|\frac{p_n}{W_n}\right|\sqrt{\mu'}\,d\theta + \int_0^{2\pi}\left|\frac{p_n}{W_n}\right|^2 d\mu = \left\| \left|\frac{p_n}{W_n}\right| - \frac{1}{\sqrt{\mu'}}\right\|^2_{L^2(\mu)}.
\end{aligned}$$

Hence,

$$\left\| \left|\frac{\varphi_n(\mu_n;\cdot)}{W_n}\right|^2 - \frac{1}{\mu'}\right\|_{L^1(\mu)} \leqslant 2\left\| \left|\frac{p_n}{W_n}\right| - \frac{1}{\sqrt{\mu'}}\right\|_{L^2(\mu)}.$$

This proves (2).

Now, let us show (4). The set of continuous functions is dense in $L^2(\mu)$. The function $1/\sqrt{\mu'}$ belongs to $L^2(\mu)$ and is nonnegative, hence it can be approximated in the metric of this space by positive continuous functions. In turn, from Lemma 2 every positive continuous function on Γ can be approximated by functions of the form $\left|\frac{p_n(z)}{W_n(z)}\right|$ (with $p_n \in \Pi_n$) and the proof is concluded. □

2.2. Proof of Theorem 2

Recall that, for every $n \geqslant 0$, we have

$$\begin{aligned}\Phi_{n+1}(\mu; z) &= z\Phi_n(\mu; z) + \Phi_{n+1}(\mu; 0)\Phi_n^*(\mu; z),\\ \Phi_{n+1}^*(\mu; z) &= \Phi_n^*(\mu; z) + \overline{\Phi_{n+1}(\mu; 0)}\, z\Phi_n(\mu; z)\end{aligned} \tag{10}$$

(see, for example, [8]).

Also, we will use that, for $\zeta_1,\ \zeta_2$ in $\{z \in \mathbb{C} : |z| < 1\}$, the following inequality holds:

$$\left|\frac{\zeta_1 - \zeta_2}{1 - \overline{\zeta_1}\zeta_2}\right| \leqslant \frac{|\zeta_1| + |\zeta_2|}{1 + |\zeta_1||\zeta_2|}$$

(see [17, p. 229]). From (10) and this inequality we obtain

$$\begin{aligned}\left|\frac{\Phi_{n+1}(\mu; z)}{\Phi_{n+1}^*(\mu; z)}\right| &= \left|\frac{z\Phi_n(\mu; z) + \Phi_{n+1}(\mu; 0)\Phi_n^*(\mu; z)}{\Phi_n^*(\mu; z) + \overline{\Phi_{n+1}(\mu; 0)}\, z\Phi_n(\mu; z)}\right|\\ &= \left|\frac{z\,\dfrac{\Phi_n(\mu; z)}{\Phi_n^*(\mu; z)} + \Phi_{n+1}(\mu; 0)}{1 + \overline{\Phi_{n+1}(\mu; 0)}z\,\dfrac{\Phi_n(\mu; z)}{\Phi_n^*(\mu; z)}}\right| \leqslant \frac{|z|\left|\dfrac{\Phi_n(\mu; z)}{\Phi_n^*(\mu; z)}\right| + |\Phi_{n+1}(\mu; 0)|}{1 + |\Phi_{n+1}(\mu; 0)|\left|\dfrac{\Phi_n(\mu; z)}{\Phi_n^*(\mu; z)}\right||z|}.\end{aligned} \tag{11}$$

Now, we suppose that, for every n, z_n is a zero of $\Phi_n(\mu; z)$ in D_r. Then

$$\Phi_{n+1}(\mu; 0) = -z_{n+1}\,\frac{\Phi_n(\mu; z_{n+1})}{\Phi_n^*(\mu; z_{n+1})}$$

and (11) becomes

$$\left|\frac{\Phi_{n+1}(\mu; z)}{\Phi_{n+1}^*(\mu; z)}\right| \leqslant \frac{|z|\left|\dfrac{\Phi_n(\mu; z)}{\Phi_n^*(\mu; z)}\right| + |z_{n+1}|\left|\dfrac{\Phi_n(\mu; z_{n+1})}{\Phi_n^*(\mu; z_{n+1})}\right|}{1 + |z_{n+1}|\left|\dfrac{\Phi_n(\mu; z_{n+1})}{\Phi_n^*(\mu; z_{n+1})}\right|\left|\dfrac{\Phi_n(\mu; z)}{\Phi_n^*(\mu; z)}\right||z|}.$$

Since $|z_{n+1}| \leqslant r$,

$$\left\|\frac{\Phi_{n+1}(\mu; z)}{\Phi_{n+1}^*(\mu; z)}\right\|_{D_r} \leqslant 2r\left\|\frac{\Phi_n(\mu; z)}{\Phi_n^*(\mu; z)}\right\|_{D_r}.$$

Thus

$$\limsup_{n\to\infty} \frac{\left\| \dfrac{\Phi_{n+1}(\mu; z)}{\Phi^*_{n+1}(\mu; z)} \right\|_{D_r}}{\left\| \dfrac{\Phi_n(\mu; z)}{\Phi^*_n(\mu; z)} \right\|_{D_r}} \leqslant 2r;$$

hence,

$$\limsup_{n\to\infty} |\Phi_n(\mu; 0)|^{1/n} \leqslant \limsup_{n\to\infty} \left\| \frac{\Phi_n(\mu; z)}{\Phi^*_n(\mu; z)} \right\|_{D_r}^{1/n} \leqslant 2r$$

and the proof is concluded. □

3. Some results on varying measures

Using Lemma 2 on rational approximation, we can easily prove the following result which is a cornerstone in the theory of orthogonal polynomials with respect to varying measures. The original proof of this result can be found in [9, Theorem 1].

Corollary 3. *If* $\lim_{n\to\infty} W_n(z)/W_n^*(z) = 0$ *uniformly on compact subsets of* $\{z : |z| < 1\}$, *then, for every continuous function* f *on* Γ, *we have*

$$\lim_{n\to\infty} \int f(z) \frac{|W_n(z)|^2}{|\varphi_n(\mu_n; z)|^2}\, d\theta = \int f(z)\, d\mu(\theta), \quad z = e^{i\theta}.$$

Proof. Let f be a continuous function on Γ. Taking into account Lemma 2, there exist two sequences of polynomials $\{p_n(z)\}$, $\{q_n(z)\}$ with $\deg p_n \leqslant n$ and $\deg q_n \leqslant n$, such that

$$\lim_{n\to\infty} \frac{p_n(z) + q_n(\frac{1}{z})}{|W_n(z)|^2} = f(z), \quad z = e^{i\theta}$$

uniformly on Γ. Also, let us note that $|W_n(z)|^2 = W_n(z)\overline{W_n(z)}$ and $p_n(z) + q_n(\frac{1}{z})$ are linear combinations of z^j ($j = 0, \pm 1, \ldots, \pm n$). Then, using (1) for $z = e^{i\theta}$, we have

$$\begin{aligned} &\left| \int f(z) \frac{|W_n(z)|^2}{|\varphi_n(\mu_n; z)|^2}\, d\theta - \int f(z)\, d\mu(\theta) \right| \\ &\quad \leqslant \int \left| f(z) - \frac{p_n(z) + q_n(\frac{1}{z})}{|W_n(z)|^2} \right| \frac{|W_n(z)|^2}{|\varphi_n(\mu_n; z)|^2}\, d\theta \\ &\quad\quad + \int \left| \frac{p_n(z) + q_n(\frac{1}{z})}{|W_n(z)|^2} - f(z) \right| d\mu(\theta) \end{aligned}$$

$$\leqslant \sup_{z\in\Gamma}\left|f(z)-\frac{p_n(z)+q_n(\frac{1}{z})}{|W_n(z)|^2}\right|\left(\int \frac{|W_n(z)|^2}{|\varphi_n(\mu_n;z)|^2}\,d\theta+\int d\mu(\theta)\right)$$
$$=2\sup_{z\in\Gamma}\left|f(z)-\frac{p_n(z)+q_n(\frac{1}{z})}{|W_n(z)|^2}\right|\int d\mu(\theta)$$

and the proof easily follows. □

To finish this section, let us remark that, using the arguments employed in [14], we can obtain the following result. The statement of this corollary is contained in [10, Lemma 2] where, instead of $\lim_{n\to\infty} W_n(z)/W_n^*(z)=0$, a weaker Carleman-type condition in terms of generalized moments is imposed. Our approach considerably simplifies the proof in this restricted situation.

Corollary 4. *If* $\mu'>0$ *a.e. in* $[0,2\pi)$, *and* $\lim_{n\to\infty} W_n(z)/W_n^*(z)=0$ *uniformly on compact subsets of* $\{z:|z|<1\}$, *then*

$$\lim_{n\to\infty}\int_0^{2\pi}\left|\frac{|\varphi_n(\mu_n;z)|\sqrt{\mu'(\theta)}}{|W_n(z)|}-1\right|^2 d\theta=0,\qquad z=e^{i\theta}.$$

4. Concerning Theorem 2

4.1. An alternative description and consequences

It is worth noticing the following alternative description of Theorem 2:

Corollary 5. *Let* $R=\limsup_{n\to\infty}|\Phi_n(\mu;0)|^{1/n}$ *and* $0\leqslant r<R/2$. *Then, there exists a sequence of indices* $\Lambda\subseteq\mathbb{N}$ *such that the polynomials* $\{\Phi_n(\mu;z):n\in\Lambda\}$ *have no zeros in* $\{z:|z|\leqslant r\}$.

Also, let us reproduce the following theorem from Nevai and Totik. We will see that its combination with Corollary 5 will have nice consequences.

Lemma 3 (*Nevai and Totik [12, Theorem 1]*). *For every n, let* $\{z_{n,k}\}_{k=1}^n$ *be the zeros of* $\Phi_n(\mu;z)$ *ordered in such a way that* $|z_{n,k+1}|\leqslant|z_{n,k}|$. *Then*

$$R=\limsup_{n\to\infty}|\Phi_n(\mu;0)|^{1/n}=\inf_k\limsup_{n\to\infty}|z_{n,k}|.$$

In particular, the number of points in $\{z_{n,k}\}_{k=1}^n\cap\{z:|z|\geqslant r\}$ *is bounded as a function of n for every* $r>R$.

Let us denote

$$Z_{\mathrm{SL}}(\mu)=\left\{z_0\in D_1:\lim_n \operatorname{dist}\,(z_0,\{\text{zeros of }\Phi_n(\mu;\cdot)\})=0\right\}.$$

In [16, § 1.7], Simon asks what type of set can Z_{SL} be. Following this line, Totik[3] asked if $Z_{\mathrm{SL}} = D_1$ could be possible. Combining Corollary 5 and Lemma 3 we obtain that

$$\{z : r_1 < |z| < r_2\} \not\subset Z_{\mathrm{SL}}$$

for every $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 < r_2/2$.

Furthermore, using Corollary 5 we can prove the following result:

Corollary 6. *If* $\lim_{n\to\infty} \Phi_n(\mu; 0) = 0$, *then there exists a sequence* $\Lambda \subseteq \mathbb{N}$ *such that*

$$\lim_{n\in\Lambda} \sum_{j=1}^{n} (1 - |z_{n,j}|) = \infty.$$

Proof. Let us divide the proof in two cases.

(i) *Case when* $\limsup_{n\to\infty} |\Phi_n(\mu; 0)|^{1/n} < 1$: By Nevai–Totik's theorem (Lemma 3 in this paper), the number of zeros of Φ_n as a function of n is bounded outside the circle $\{z : |z| \leqslant r\}$ for every r such that $1 > r > \limsup_{n\to\infty} |\Phi_n(\mu; 0)|^{1/n}$. Consequently, $\lim_{n\to\infty} \sum_{j=1}^{n} (1 - |z_{n,j}|) = \infty$.

(ii) *Case when* $\limsup_{n\to\infty} |\Phi_n(\mu; 0)|^{1/n} = 1$: Let us take $r < \frac{1}{2}$. By Corollary 5, there exists a sequence of indices $\Lambda \subseteq \mathbb{N}$ such that $\{\Phi_n(\mu; z) : n \in \Lambda\}$ has no zeros in $\{z : |z| \leqslant r\}$. Then, for any numbers $\xi_{n,j} \in \mathbb{C}$ ($n \in \Lambda$, $j = 1, \ldots, n$) with $|\xi_{n,j}| = |z_{n,j}|$, we have $\xi_{n,j} \notin \{z : |z| \leqslant r\}$ and $\lim_{n\in\Lambda} \prod_{j=1}^{n} |\xi_{n,j}| = \lim_{n\in\Lambda} \prod_{j=1}^{n} |z_{n,j}| = 0$. Thus,

$$\Psi_n(z) = \prod_{j=1}^{n} \frac{z - \xi_{n,j}}{1 - \overline{\xi_{n,j}}\, z}, \quad n \in \Lambda,$$

is an uniformly bounded sequence of functions and $\Psi_n(z) \neq 0$ when $|z| \leqslant r$. On the other hand, every limit function of the family $\{\Psi_n : n \in \Lambda\}$ vanishes at $z = 0$; hence, by Hurwitz's theorem, it vanishes identically. Therefore, $\{\Psi_n : n \in \Lambda\}$ converges uniformly to 0 on compact subsets of $\{z : |z| < 1\}$. Now, applying [17, Theorem 9, p. 247] it follows that $\lim_{n\in\Lambda} \sum_{j=1}^{n} (1 - |z_{n,j}|) = \infty$. □

Remark 2. With respect to the reciprocal of Corollary 6, let us note the following. Since

$$|\Phi_n(\mu; 0)| = \prod_{j=1}^{n} |z_{n,j}| = \exp\left(\sum_{j=1}^{n} \log |z_{n,j}|\right) \leqslant \exp\left(-\sum_{j=1}^{n} (1 - |z_{n,j}|)\right),$$

if $\lim_{n\in\Lambda} \sum_{j=1}^{n} (1 - |z_{n,j}|) = \infty$, it follows that $\lim_{n\in\Lambda} \Phi_n(\mu; 0) = 0$.

4.2. Limiting distribution of zeros

Theorem 2 gives some information about the limiting distribution of the zeros of the orthogonal polynomials $\Phi_n(\mu; z)$. Let us see why.

[3] In the V International Meeting on Approximation, held at Ubeda (Spain) on June 9–14, 2004.

As usual, given p_n a polynomial of degree n, let $\nu(p_n) = (1/n) \sum_{\{\zeta : p_n(\zeta)=0\}} \delta_\zeta$ denote the normalized zero counting measure of p_n. For $\lambda > 0$, let ν_λ be the arc-measure $(2\pi\lambda)^{-1}\, d\theta$ on the circle $C_\lambda = \{z \in \mathbb{C} : |z| = \lambda\}$. Finally, let us take

$$\rho = \limsup_{n\to\infty} |\Phi_n(\mu; 0)|^{1/n} \leqslant \min\{1,\ 2r\}.$$

(i) *Case* $\rho < 1$ (*of course, this always happens if* $r < \frac{1}{2}$): From [15, Theorem 5.3], we know that every weakly convergent subsequence of $\{\nu(\Phi_n(\mu; \cdot))\}$ converges to ν_λ for some $\lambda \leqslant 2r$. In other words, let $\Lambda \subseteq \mathbb{N}$ be a sequence for which $\lim_{n\in\Lambda} |\Phi_n(\mu; 0)|^{1/n} = \rho$. Then, the support of the zero distribution of the orthogonal polynomials $\{\Phi_n(\mu; z)\}_{n\in\Lambda}$ is a circle of radius at most $2r$.

(ii) *Case* $\rho = 1$ (*observe that this requires that* $r \geqslant \frac{1}{2}$): As in Lemma 3, for n fixed, let us consider the zeros $\{z_{n,k}\}_{k=1}^{n}$ of $\Phi_n(\mu; z)$ ordered with $|z_{n,k+1}| \leqslant |z_{n,k}|$. Then, we deduce that $\limsup_{n\to\infty} \left|z_{n,k}\right| = 1$ for every $k \geqslant 1$. So, there exists $\Lambda = \Lambda(k) \subseteq \mathbb{N}$ such that $\lim_{n\in\Lambda} \left|z_{n,k}\right| = 1$. We can obtain additional information in the special case when $\lim_{n\to\infty} |\Phi_n(\mu; 0)| = \alpha$ exists (see [1, Proposition 2.1] for items (a) and (b) below):

(a) If $\alpha \in (0, 1)$, then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$z_{n,k} \in \{z : \alpha - \varepsilon < |z| < 1\}, \quad 1 \leqslant k \leqslant n,$$

when $n \geqslant n_0$.

(b) If $\alpha = 1$, all the zeros of $\Phi_n(\mu; z)$ are, for n large enough, arbitrarily close to the unit circle Γ.

(c) If $\alpha = 0$ then $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left|\Phi_j(\mu; 0)\right| = 0$. In this way [15, Theorem 5.3] states the weak convergence

$$\nu(\Phi_n(\mu; \cdot)) \xrightarrow{*} \eta,$$

where η is a measure whose balayage on Γ is $(2\pi)^{-1}\, d\theta$.

4.3. Graphics of zeros

Finally, we include some graphics that indicate that the bound (5) may be sharp. The orthogonal polynomials are generated with the following method:

Given $r < 1$, *for* $N \in \mathbb{N}$ *large enough, let us take* $t_j = r\exp((2j+1)\pi i/N)$, $j = 0, 1, \ldots, N-1$. *Then we obtain* z_k *and the orthogonal polynomials* Φ_k *as follows. For* $k = 0$, *we take* $\Phi_1(z) = 1$. *For* $k = 1$, *we set*

$$z_1 = t_0,$$
$$\Phi_1(z) = z - z_1.$$

From $k = 2$ *to N, we choose* $z_k \in \{t_j : j = 0, \ldots, N-1\} \setminus \{z_{k-1}\}$ *such that*

$$\left|\frac{\Phi_{k-1}(z_k)}{\Phi_{k-1}^*(z_k)}\right| = \max\left\{\left|\frac{\Phi_{k-1}(t)}{\Phi_{k-1}^*(t)}\right| : t = t_j,\ j = 0, \ldots, N-1,\ t \neq z_{k-1}\right\},$$

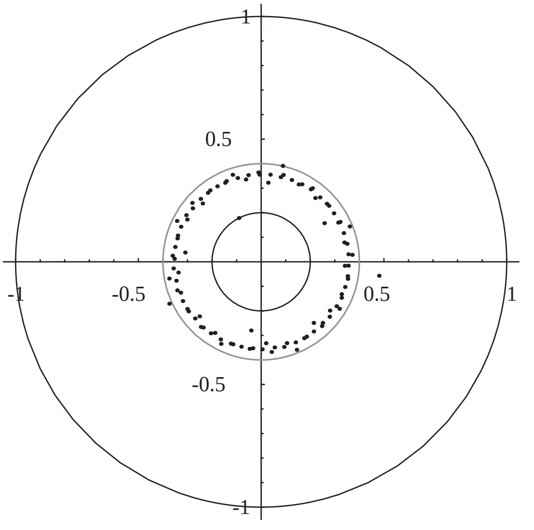

Fig. 1. Zeros when $r = 0.2$, $N = 100$.

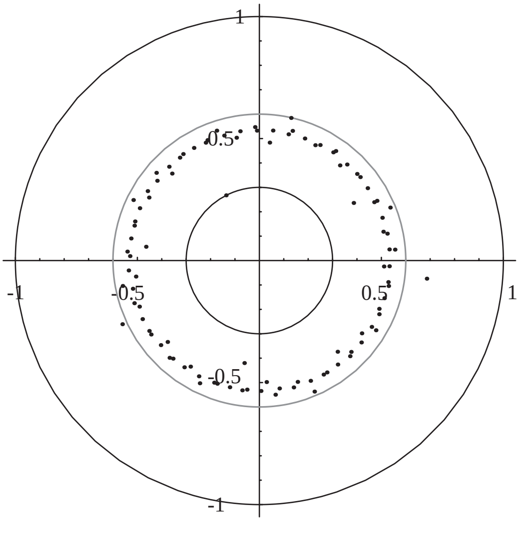

Fig. 2. Zeros when $r = 0.3$, $N = 100$.

and we make

$$\Phi_k(z) = z\Phi_{k-1}(z) - z_k \, \frac{\Phi_{k-1}(z_k)}{\Phi^*_{k-1}(z_k)} \, \Phi^*_{k-1}(z).$$

So, $\Phi_k(z_k) = 0$. *Moreover, taking into account Verblunsky's theorem* (*see* [16]), *there exists a measure* $\mu \in \mathcal{M}$ *such that* $\{\Phi_n = \Phi_n(\mu; \cdot) : n = 0, 1, 2, \ldots\}$.

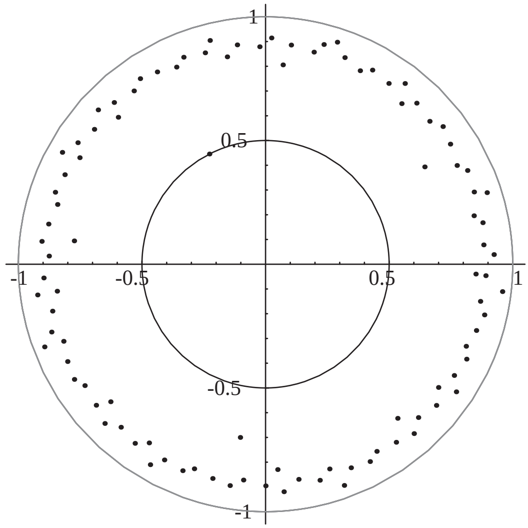

Fig. 3. Zeros when $r = 0.5$, $N = 100$.

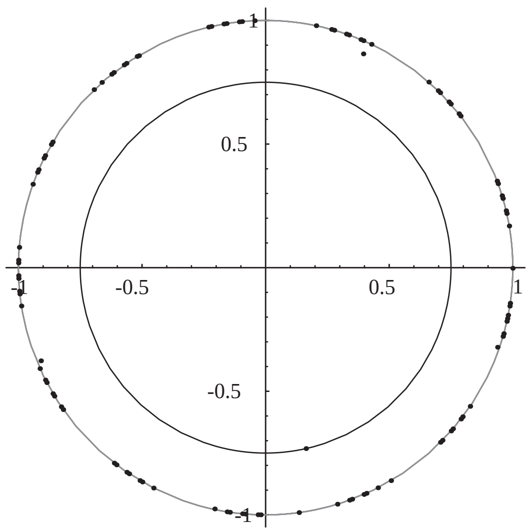

Fig. 4. Zeros when $r = 0.75$, $N = 100$.

We illustrate this situation with several figures coming from numerical experiments using the described algorithm. In each of the Figs. 1–4, we have a different value for the radius, i.e., $r = 0.2, 0.3, 0.5$, and 0.75, respectively, and we have plotted the zeros of Φ_{100}. In accordance with the described method, a zero in the circle of radius r has been fixed; in the figures, observe this circle and the zero. We can see that almost all the other zeros are near the circle of radius $\min\{1, 2r\}$ (in the figures, this circle is represented with gray color). Moreover, we have computed the values for $|\Phi_{100}(0)|^{1/100}$; they are 0.354, 0.531, 0.881, and 0.996, respectively. From [15] (see also [11]), this means that in these cases the bound (5) seems to be exact.

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